

Correlated and Coarse Equilibria of Single-Item Auctions

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Abstract

We study correlated equilibria and coarse equilibria of simple first-price single-item auctions in the simplest auction model of full information. Nash equilibria are known to always yield full efficiency and a revenue that is at least the second-highest value. We prove that the same is true for all *correlated equilibria*, even those in which agents overbid – i.e., bid above their values.

Coarse equilibria, in contrast, may yield lower efficiency and revenue. We show that the revenue can be as low as 26% of the second-highest value in a coarse equilibrium, even if agents are assumed not to overbid, and this is tight. We also show that when players do not overbid, the worst-case bound on social welfare at coarse equilibrium improves from 63% of the highest value to 81%, and this bound is tight as well.

1 Introduction

A very basic tenet of economic theory is to analyze strategic situations such as games or markets *in equilibrium*. The logic being that systems will typically reach an equilibrium point, following some dynamic, a dynamic that may be difficult to understand or analyze. Of course, in order for the equilibrium concept to be predictive, it must correspond to outcomes of the types of dynamics we consider possible. In Game Theory, the leading equilibrium concept is a Nash equilibrium.

In Algorithmic Game Theory, Nash equilibrium is not the only notion of equilibrium that is considered. On the one hand, it is typically computationally-hard to find a Nash equilibrium, and so it is questionable whether a Nash equilibrium can be viewed as a reasonable prediction of an outcome of a game. In contrast, there are a host of natural “learning-like” dynamics that converge to more general notions of equilibria, specifically to *correlated equilibria* or to the even more general *coarse equilibria*¹ which often seem to be more natural predications than Nash equilibria. It is thus common to consider also these more general notions of equilibrium in scenarios studied in Algorithmic Mechanism Design.

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¹Sometimes called “coarse correlated equilibria” [19] or “Hannan consistent” [10].

This extension of our concept of the class of possible equilibria has a bright side and a dark side. On the negative side, if we accept that one of these generalized equilibria notions is a possible outcome, then we need to ensure that all such equilibria produce whatever result is desired by us (in terms of “Price of Anarchy”, we can get worse bounds as we need to take the worst case performance over a wider set of equilibria.) On the positive side, in cases where we can control the equilibrium reached (e.g. by coding specific dynamics into software), we may take advantage of the extra flexibility to obtain better equilibria points (in terms of “Price of Stability”, we may get better bounds, as we can choose an equilibrium within a wider class).

In this paper we study correlated equilibria and coarse equilibria in the simplest auction model: a full-information first-price auction of a single item. This is the simplest instance in the class of simultaneous auctions studied in a line of work that has received much attention lately [3, 1, 12, 9, 18]. Here too the literature is concerned both about the difficulty of reaching equilibria [2, 7, 5, 6] and about the additional loss of efficiency or revenue in these generalized types of equilibria². A loss of efficiency here corresponds to mis-allocation of the item (i.e. the winner not being the bidder with the highest value), while a loss of revenue may also be viewed as a type of implicit self-stabilizing collusion between the bidders [14].

For concreteness, consider the case where Alice has value 1 for the item and Bob has value 2 where the values are common knowledge and they are participating in a first price auction. In this game the strategy space of each bidder is the set of possible bids (non-negative numbers), and the outcome from a pair of bids a by Alice and b by Bob is that Alice wins whenever $a > b$ and pays a (so her utility is $1 - a$ and Bob’s utility is 0) and Bob wins whenever $b > a$ and pays b (so his utility is $2 - b$ and Alice’s utility is 0). For simplicity, let us assume that ties are broken in favor of Bob, i.e. that he wins whenever $b \geq a$.

It is quite easy to analyze the pure Nash equilibria of this game: for every value of $1 \leq v \leq 2$ there exists an equilibrium where both Alice and Bob bid the same value v , and Bob wins the tie. In the general case of first price auctions, the price v may be anything between the first price and the second price in the auction.³ While non-trivial, it is also not difficult to analyze the mixed Nash equilibria of this game, where it turns out that every mixed Nash equilibrium is outcome-equivalent to a pure Nash equilibrium [4]. “Outcome equivalent” means that we get the same distribution of the identity of the winner and of his payment. In particular, all mixed (or pure) Nash equilibria of a full-information first price auction attain perfect social welfare (i.e. the player with highest value always wins) and have revenue that is bounded below by the second highest value in the auction (and from above by the highest value).

What about correlated equilibria and coarse equilibria? It is easy to see that correlated equilibria

²For example, while pure Nash equilibria of simultaneous first-price auctions are known to be fully efficient, mixed Nash equilibria may not [12].

³This requires that the player with the highest value – Bob in our example – wins the tie; otherwise no pure equilibrium exists but arbitrarily close ϵ -equilibria do.

give a somewhat richer class of outcomes, since certainly a single correlated equilibrium can mix between several pure equilibria. Our first result shows that this is all that can be obtained, so in particular, correlated equilibria also yield perfect social welfare and a revenue that is at least the second highest value.

Theorem: Every correlated equilibrium of a first-price auction is outcome-equivalent to a mixture of pure Nash equilibria.

In [14] a similar theorem was proved for the special case of symmetric bidders, even in Bayesian settings. Whether or not correlated equilibria can be richer in non-symmetric Bayesian settings remains open⁴. There are several other known cases of games where correlated equilibria cannot improve upon Nash equilibria (see [15] and references therein). In [8] it was shown, in a more general setting than the one described here, that there is a *unique* correlated equilibrium if one eliminates *weakly dominated strategies*. That is, if no player ever bids above their value. One can indeed verify that the only correlated equilibrium satisfying this constraint is the pure Nash equilibrium in which both agents bid the second-highest value.

We then turn our attention to coarse equilibria, and it turns out that a wider set of outcomes becomes possible. In [17] a coarse equilibrium is exhibited in a two-player single-item auction that is not outcome-equivalent to a mixture of pure auctions. In fact, its welfare is only $1 - 1/e = 0.632120\dots$ fraction of the optimal. This matches the general Price of Anarchy upper bound given in [18] (established via the smoothness technique [16, 18]), which applies even to general multi-item simultaneous auctions with XOS bidders. For multi-item simultaneous auctions, this bound of $1 - 1/e$ is tight even with respect to Nash equilibria [4], but for single-item auctions, as we have seen, it is only tight for coarse equilibria and not for correlated or for Nash equilibria.

The example that attains this low welfare has the undesirable property that it uses weakly dominated strategies. I.e. in this example, the support of the coarse equilibrium contains strategies where one of the players bids above his value. The natural question at this point is whether there exists other coarse equilibria that do not use over-bidding. Consider the following example, in Table 1, of a coarse equilibrium, where ϵ is some small enough constant (e.g. $\epsilon = 10^{-4}$).

One may directly verify that this is indeed a coarse equilibrium.⁵ This finite equilibrium allocates the item to Alice sometimes, and so the social welfare that it reaches is not perfect! Also notice that the winner pays at most 1 for the item, but sometimes pays strictly less than 1, and

⁴Note, however, that in asymmetric Bayesian settings, even in (Bayesian) Nash equilibria, the winner is not necessarily the bidder with the highest valuation [13].

⁵Ignoring $O(\epsilon)$ terms, at equilibrium we have: For Alice: $u_A = 0.02 * 1 + 0.02 * 0.9 = 0.038$ while deviating to 0 would yield utility 0.02, deviating to 0.1 yield utility 0.036, deviating to 0.5 yield utility 0.035, deviating to 0.8 yield utility 0.036, deviating to 0.9 yield utility 0.37 and deviating to 1 yield utility 0. For Bob we have $u_B = 0.03 * 1.5 + 0.11 * 1.2 + 0.19 * 1.1 + 0.63 * 1 = 1.016$, but deviating to 1 would give utility 1, and deviating to anything below 1 would lose with probability of at least 63% leading to utility that is certainly less than 1.

Table 1: A coarse equilibrium in an auction with $v_{Alice} = 1$, $v_{Bob} = 2$

Probability	Alice's Bid	Bob's Bid
2%	ϵ	0
2%	$0.1 + \epsilon$	0.1
3%	$0.5 - \epsilon$	0.5
11%	$0.8 - \epsilon$	0.8
19%	$0.9 - \epsilon$	0.9
63%	$1 - \epsilon$	1

thus the revenue is strictly smaller than the second price!

At this point, the natural question is what is the lowest welfare possible in a coarse equilibrium where no player overbids. We might term this ratio “PoUA” – “the price of *undominated* anarchy”. We show that indeed insisting that players never overbid ensures a significantly higher share of welfare, and provide tight bounds for it. To the best of our knowledge, this is the first indication that a no overbidding restriction improves worst case guarantees in first-price auctions.⁶

Theorem: In every coarse equilibrium of a single-item first-price auction where players never bid above their value, the social welfare is at least 0.813559... fraction of the optimal.

Theorem: There exists a single-item first-price auction with two players that has a coarse equilibrium where players never bid above their value, with social welfare that is 0.813559... fraction of the optimal.

We then focus our attention on the revenue of the auction. While Nash equilibria and correlated equilibria always yield revenue that is at least the second highest value, our example above has shown that coarse equilibrium may yield lower revenue. We ask how low may this revenue be, and provide a tight bound:

Theorem: In every coarse equilibrium of any single-item first-price auction, the revenue is at least $1 - 2/e = 0.264241...$ fraction of the second highest value.

Theorem: There exists a single-item first-price auction with two players that has a coarse equilibrium where players never bid above their value whose revenue is only $1 - 2/e = 0.264241...$ fraction of the second highest value.

Notice that here we get the same bound whether or not players can bid above their value. This lower bound is obtained in a symmetric instance (i.e., where the two players have the same value).

⁶In contrast, for multi-item simultaneous auctions, the $1 - 1/e$ bound is tight for XOS valuations even without overbidding [3].

We remark that in large symmetric instances the revenue approaches the value of the players. We also show that as the gap between the highest value and the second highest value increases, the revenue approaches the second highest value. This is in contrast to social welfare, where noted above the inefficiency may persist even when the gap in the players' values is arbitrarily large.

2 Preliminaries

We will focus on an auction with n players and a single item for sale. Player i has value v_i for the item, and we index the players so that $v_1 \geq v_2 \geq \dots \geq v_n$.

The auction proceeds as follows: the players simultaneously submit real-valued bids, $\mathbf{x} = (x_1, \dots, x_n)$. Ties are broken according to a fixed tie-breaking function, which maps the (maximal) bids to a winner. Player i wins when $x_i \geq x_j$ for all j and the tie at value x_i (if any) is broken in favor of player i , which we denote by $x_i \succsim \mathbf{x}_{-i}$. The winner pays his or her bid.

Given a joint distribution D over the bids of the players, the expected payment of player i is $E_{\mathbf{x} \sim D}[x_i \cdot 1_{x_i \succsim \mathbf{x}_{-i}}]$, so his expected utility is $u_i = v_i \cdot Pr_{\mathbf{x} \sim D}[x_i \succsim \mathbf{x}_{-i}] - E_{\mathbf{x} \sim D}[x_i \cdot 1_{x_i \succsim \mathbf{x}_{-i}}]$.

We will study correlated and coarse correlated equilibria of the auction. The following definitions are tailored to our auction setting; a more general definition of correlated equilibria for infinite games can be found in Hart and Schmeidler [11].

Definition 2.1. *A joint distribution D over bids is a correlated equilibrium if, for every player i and every (measurable) deviation function $b_i: \mathbb{R} \rightarrow \mathbb{R}$ of player i , it holds that*

$$v_i \cdot Pr_{\mathbf{x} \sim D}[x_i \succsim \mathbf{x}_{-i}] - E_{\mathbf{x} \sim D}[x_i \cdot 1_{x_i \succsim \mathbf{x}_{-i}}] \geq v_i \cdot Pr_{\mathbf{x} \sim D}[b_i(x_i) \succsim \mathbf{x}_{-i}] - E_{\mathbf{x} \sim D}[b_i(x_i) \cdot 1_{b_i(x_i) \succsim \mathbf{x}_{-i}}].$$

Definition 2.2. *A joint distribution D over bids is a coarse correlated equilibrium (or coarse equilibrium for short) if, for every player i and for every unilateral deviation $x'_i \in \mathbb{R}$ of player i , it holds that*

$$v_i \cdot Pr_{\mathbf{x} \sim D}[x_i \succsim \mathbf{x}_{-i}] - E_{\mathbf{x} \sim D}[x_i \cdot 1_{x_i \succsim \mathbf{x}_{-i}}] \geq v_i \cdot Pr_{\mathbf{x} \sim D}[x'_i \succsim \mathbf{x}_{-i}] - E_{\mathbf{x} \sim D}[x'_i \cdot 1_{x'_i \succsim \mathbf{x}_{-i}}].$$

That is, correlated equilibria are immune to deviations that can condition on the recommended bid, whereas coarse equilibria need only be immune to unconditional deviations (i.e., constant bidding functions).

2.1 Tie Breaking

Before we continue, a word about tie-breaking is in order. All of our theorems that claim something for all equilibria will hold for *every* tie breaking rule. In our constructions, we will allow ourselves to choose a tie breaking rule to our liking. Note however that if the tie breaking rule is not to the

reader's liking, in a joint distribution over \mathbf{x} we can always avoid any dependence on it by increasing one of the maximal bids by ϵ , which would give us an ϵ -equilibrium (rather than an exact one). Thus every construction of an equilibrium that we provide using a particular tie-breaking rule immediately implies also an ϵ -equilibrium for any tie breaking rule and any $\epsilon > 0$.

2.2 The Distribution on the Winning Price

When analysing revenue and welfare in an equilibrium, it will be most convenient to consider the single-dimensional distribution on the winning price, i.e., on $\max_i \{x_i\}$. We will denote the cumulative distribution on the winning price by F . The revenue can be easily expressed in terms of F as $Revenue = E_{x \sim F}[x] = \int (1 - F(x))dx$ (where the integration is over the support of the distribution.)

The starting point for our analysis of *coarse equilibria* is the following. Suppose there are $n = 2$ players, say Alice and Bob, with values 1 and $v \leq 1$ respectively. If Alice chooses to deviate from the equilibrium to some fixed bid x , then her utility will be $(1 - x) \cdot F_{Bob}(x) \geq (1 - x) \cdot F(x)$ (this expression ignores the possibility of a tie), where F_{Bob} is the cumulative distribution of Bob's bid, which is certainly stochastically dominated by the cumulative distribution on the winning bid. In our constructions we will typically have both Alice and Bob always bidding the same value $x = y$ (where this joint value is distributed according to F), and thus will have $F_{Bob} = F$.

Denoting Alice's utility at equilibrium by α , a necessary condition that this deviation is not profitable is thus $\alpha \geq (1 - x) \cdot F(x)$, i.e. that $F(x) \leq \alpha/(1 - x)$. Similarly for Bob we must have $F(x) \leq \beta/(v - x)$, where β is Bob's utility at equilibrium. Thus if F corresponds to a coarse equilibrium then it must be stochastically dominated by the minimum of these two expressions.

The following simple calculation states the closed form expression for the revenue of a distribution of these forms.

Lemma 2.1. *Let the cumulative distribution function $G = G_{a,b}$ be defined by $G(x) = a/(b - x)$ for $0 \leq x \leq b - a$. Then, $E_G[x] = b - a + a \ln(a/b)$.*

3 Correlated equilibrium

The goal of this section is to establish that every correlated equilibrium of a first-price auction is outcome-equivalent to a mixture of pure Nash equilibria. We begin by showing that the winning bid is never lower than the second-highest of the players' values.

Lemma 3.1. *For every correlated equilibrium D , $Pr_{\mathbf{x} \sim D}[\max_i \{x_i\} < v_2] = 0$*

Proof. Assume otherwise. Let $S = \{p | Pr_{\mathbf{x} \sim D}[\max_i \{x_i\} < p] > 0\}$, and let $p^* = \inf(S) < v_2$. Fix

some $p^* < p < (p^* + v_2)/2$, for which we thus have $Pr_{\mathbf{x} \sim D}[\max_i \{x_i\} < p] = \delta > 0$.

Consider players 1 and 2. One of the two players must be winning with probability of at most $\delta/2$ (when $\max_i \{x_i\} < p$), say player $j \in \{1, 2\}$. I.e., $Pr_{\mathbf{x} \sim D}[p > x_j \succsim \mathbf{x}_{-j}] \geq \delta/2$. (recall that $x_j \succsim \mathbf{x}_{-j}$ means that either x_j is strictly larger than the other bids, or that it is weakly larger and tie at that value is broken in favor of player j).

Now let's consider the following deviation function b_j for player j . Choose $\epsilon > 0$ so that $p + 2\epsilon < (p^* + v_2)/2$ (which is possible by the choice of p above). Then define $b_j(x_j) = p + \epsilon$ for all $x_j \leq p$, and $b_j(x_j) = x_j$ for all $x_j > p$. That is, when being told any value $x_j \leq p$, player j bids instead $p + \epsilon$. On the up side, this will certainly win all the cases where $\max_i \{x_i\} < p$, increasing the probability of winning by at least $\delta/2$ and thus increasing his utility by at least $\delta \cdot (v_2 - p - \epsilon)/2$. On the down side, player j now pays $p + \epsilon$ when he wins rather than his original bid x_j . Since player j never won when $x_j < p^*$ (as $Pr_{\mathbf{x} \sim D}[\max_i \{x_i\} < p^*] = 0$, by definition of p^*) he pays at most $(p - p^*) + \epsilon$ more whenever he wins, which happened with probability of at most $\delta/2$. Thus the down side of player j 's utility from deviation is at most $\delta \cdot (p - p^* + \epsilon)/2$. So the deviation is profitable whenever $\delta \cdot (v_2 - p - \epsilon)/2 > \delta \cdot (p - p^* + \epsilon)/2$ which is the case due to our choice of ϵ . \square

We next show that only the players with the highest value can ever win in a correlated equilibrium.

Lemma 3.2. *For every correlated equilibrium D , and for any player i such that $v_i < v_1$, $Pr_{\mathbf{x} \sim D}[x_i \succsim \mathbf{x}_{-i}] = 0$.*

Proof. By the previous lemma, no one ever wins with a price that is strictly less than v_2 . On the other hand, if any player $i > 1$ ever wins with a price that is strictly more than v_i , their utility will be negative, making a deviation to a bid 0 profitable. This immediately implies the desired result if $v_2 = v_1$, so from this point onward we will assume $v_2 < v_1$.

The only case we further need to consider is if some player $i \geq 2$ with $v_i = v_2$ wins at price exactly v_2 , say with some probability $\delta > 0$. But then player 1 would prefer to deviate from any $x_1 \leq v_2$ to $v_2 + \epsilon$, gaining utility of at least $\delta(v_1 - v_2 - \epsilon)$ due to winning all cases in which $\max_i \{x_i\} \leq v_2$, and losing at most ϵ due to the additional payment (since player 1 never pays less than v_2 when she wins). Choosing ϵ small enough, this deviation becomes profitable. \square

In conclusion we have a complete characterization of correlated equilibria in terms of their outcomes. Clearly every mixture of pure equilibria is a correlated equilibrium, and this turns out to be all that is possible:

Theorem 3.3. *Every correlated equilibrium of the single-item first-price auction is equivalent (in terms of winning probabilities and payments) to a mixture of pure equilibria (where Alice always wins the ties).*

Proof. First suppose $v_1 > v_2$. Since by the previous lemma player 1 always wins, and never pays less than v_2 , she must always bid at least v_2 . Clearly player 1 can never bid more than v_1 since that will give her negative utility (as she does always win). Thus player 1's bid x_1 is supported on the interval $[v_2, v_1]$ and she always wins. The outcome is thus equivalent to that of a similar distribution on the pure equilibria in which all players bid $x \in [v_2, v_1]$ (with player 1 winning the ties).

If $v_1 = v_2$ then only the maximum-valued players ever win, and the winner always pays at least v_1 . The utility of every player is therefore exactly 0. The outcome is thus equivalent to a similar distribution on the pure equilibria, in which all players bid v_1 and ties are broken in favor of the appropriate maximum-value player. \square

Our characterization also implies that the revenue of the auctioneer is always at least the second highest value.

4 Price of Undominated Anarchy

The following theorem shows that if players do not over bid, the welfare guarantee in any CCE improves from 63% to 81%.

Theorem 4.1. *In every coarse equilibrium of the single-item first price auctioneer players never bid above their value, the social welfare is at least 0.813559... fraction of the optimal.*

Proof. Let us start by normalizing the values of the players in the auction: let us call the player with highest value Alice, and normalize this value to 1, and let us call the player with second highest value Bob, so his value is $v \leq 1$. There could be other players in the auction but our analysis will ignore them. Fix a coarse equilibrium of that auction. Let us further denote Alice's utility in the equilibrium by α and Bob's utility by β . Since Bob never uses undominated strategies, he always bids at most v and thus Alice can always deviate to $v + \epsilon$ obtaining a utility of $1 - v$. hence we must have $\alpha \geq 1 - v$. Denote by F_{CE} the cumulative distribution on the price paid by the winner of the auction, so the fact that Alice does not want to deviate implies that $F_{CE}(x) \leq \alpha / (1 - x)$ for all $0 \leq x \leq 1 - \alpha$ and the fact that Bob does not want to deviate implies that $F_{CE}(x) \leq \beta / (v - x)$ for all $0 \leq x \leq v - \beta$. Thus the distribution F_{CE} stochastically dominates the following distribution whose cumulative distribution function is $F(x) = \min\{\frac{\alpha}{1-x}, \frac{\beta}{v-x}\}$ for $0 \leq x \leq \max(v - \beta, 1 - \alpha)$ (and 0 for $x \leq 0$). The revenue raised by the auction is simply the expected value of the winning price which is bounded from below by the expected value of x that is drawn according to F . Thus $Revenue \geq \int_0^1 (1 - F(x)) dx$, and a lower bound to the welfare is obtained by adding this revenue to $\alpha + \beta$. We will calculate such a lower bound, over all possible values of $\alpha \geq 1 - v$, β , and $v \leq 1$. I.e. we will show that for all possible values of α, β , and v , we have that $\alpha + \beta + \int_0^1 (1 - F(x)) dx \geq 0.813559....$

In calculating $\int_0^1 (1 - F(x))dx$ we will split into two cases. We start with the easy case, where $\beta \geq v\alpha$. This is the easy case since here $\beta/(v-x) \geq 1/(1-x)$ for all $0 \leq x \leq v$ and thus F simplifies to $F(x) = \alpha/(1-x)$ for all $0 \leq x \leq 1-\alpha$, and so our integral simplifies to $Revenue = \int_0^{1-\alpha} (1 - \alpha/(1-x))dx = 1 - \alpha + \alpha \log \alpha$. Thus a lower bound on the welfare is $\alpha + \beta + 1 - \alpha + \alpha \log \alpha$. In this case we had that $\beta \geq \alpha v \geq \alpha(1-\alpha)$ so our lower bound on welfare, over all β and v is $welfare \geq 1 + \alpha(1-\alpha) + \alpha \log \alpha$. This attains its minimum of 0.838... over all $0 \leq \alpha \leq 1$ at $\alpha = 0.203..$ (where α is the solution to the equation $2x - \log x - 2 = 0$) and so we have that for the case $\beta \geq v\alpha$ the welfare is at least 0.838... $> 0.813559....$ ⁷

We continue with the more complex case where $\beta < v\alpha$ in which case we have that $\beta/(v-x) < \alpha/(1-x)$ exactly when $x < \theta = (\alpha v - \beta)/(\alpha - \beta)$ and thus the revenue is obtained as $Revenue = \int_0^\theta (1 - \beta/(v-x))dx + \int_\theta^{1-\alpha} (1 - \alpha/(1-x))dx = \alpha \log((\alpha - \beta)/(1-v)) + \beta \log((\beta(1-v))/(v(\alpha - \beta))) + 1 - \alpha$. Our lower bound for the welfare is thus $\beta + \alpha \log((\alpha - \beta)/(1-v)) + \beta \log((\beta(1-v))/(v(\alpha - \beta))) + 1$.

Taking the derivative with respect to v , we get the expression $(\alpha v - \beta)/((1-v)v)$ which is always positive in our range and thus for every α and β , the minimum is obtained at the lowest possible value $v = 1 - \alpha$. Substituting this value of v , we get that the minimum possible welfare is the minimum of the function $\beta + \alpha \log((\alpha - \beta)/\alpha) + \beta \log((\beta\alpha)/((1-\alpha)(\alpha - \beta))) + 1$. For every $\alpha > 0$, this expression evaluates to 1 at $\beta = 0$ with a negative derivative for small β . Taking the derivative with respect to β we get $1 + \log(\alpha\beta/((1-\alpha)(\alpha - \beta)))$ which becomes 0 for $\beta = (\alpha - \alpha^2)/(e\alpha - \alpha + 1)$. We now split again into two cases. In the first case, this value of β is within the possible range, $(\alpha - \alpha^2)/(e\alpha - \alpha + 1) < \alpha v = \alpha(1 - \alpha)$ and so the minimum is obtained at this value of β , and the second case is where $(\alpha - \alpha^2)/(e\alpha - \alpha + 1) > \alpha(1 - \alpha)$ in which case the minimum is obtained at the highest possible value of $\beta = \alpha(1 - \alpha)$.

We start with the second case of $\beta = \alpha(1 - \alpha)$ for which the welfare simplifies to: $1 - \alpha^2 + \alpha + \alpha \log(\alpha)$ which attains its minimum of 0.838... at $\alpha = 0.203..$, where α is the solution of $2 - 2x + \log x = 0$. (This is exactly the same point identified above by our analysis above of the easy case of $\beta \geq \alpha v$.)

For the first case where $\beta = (\alpha - \alpha^2)/(e\alpha - \alpha + 1)$, we plug in this value of β into the expression for welfare which then simplifies to $\alpha \log((e\alpha)/((e-1)\alpha + 1)) + 1$ which attains its minimum of 0.813559... at $\alpha = 0.274322....$, where α is the solution of $((e-1)x + 1) \log((ex)/((e-1)x + 1)) + 1)/((e-1)x + 1) = 0$. \square

⁷To get an auction with these parameters we need to specify when each of the players wins in a way that will achieve these values of α and β . The following parameters yield these utilities: Alice and Bob bid the same value of x distributed according to the same F that provided the lower bound: α that is the solution of the equation $2x - \log x - 2 = 0$, $v = 1 - \alpha$ and $\beta = v\alpha$. Alice wins whenever $x = 0$ and Bob wins otherwise. Thus the probability that Alice wins is $\alpha = F(0)$ and she pays nothing, indeed obtaining utility of α . Bob wins probability $p = 1 - \alpha$ and pays the entire revenue obtaining net utility of $pv - Revenue = (1 - \alpha)(1 - \alpha) - (1 - \alpha + \alpha \log \alpha)$ which for our α is indeed $(1 - \alpha)\alpha = \beta$.

We now show that this bound is tight, by exhibiting an auction with matching welfare.

Theorem 4.2. *There exists a single-item two-player auction with player values 1 and $v \leq 1$, and a coarse equilibrium of that auction where players never bid above their values, whose social welfare matches the bound from Theorem 4.1 (0.813559...).*

Proof. Our equilibrium will take a form developed in the proof of Theorem 4.1. Call the player with value 1 Alice, and the player with value $v \leq 1$ Bob.

Guided by the proof of Theorem 4.1, we will choose a parameter α , then set

$$\beta = (\alpha - \alpha^2)/(e\alpha - \alpha + 1) \quad \text{and} \quad v = 1 - \alpha. \quad (1)$$

We will think of α and β as Alice's and Bob's utilities at equilibrium, respectively.

Define $F(x) = \min\{\frac{\alpha}{1-x}, \frac{\beta}{v-x}\}$, for $x \in [0, v]$. In the equilibrium we construct, a value will be drawn from the distribution with CDF F and both players will bid that value. Note that neither Alice nor Bob has a profitable deviation in such an equilibrium, as long as their utilities are α and β respectively. Thus, to show that an equilibrium exists for a certain choice of α , we must specify when each of the players wins so that they achieve the utilities α and β .

We will show that an equilibrium exists for all $\alpha \in [0.27, 0.28]$. This will imply the desired result, since in particular this includes the value of α for which the welfare bound from Theorem 4.1 is achieved. Recall from the proof of Theorem 4.1 that, if an equilibrium exists, its welfare will be $W = \alpha \log((e\alpha)/((e-1)\alpha + 1)) + 1$.

Write q for the solution to $W = v(1 - q) + q$, so that

$$q = \frac{W-v}{1-v}. \quad (2)$$

We first claim that if we are able to specify when Alice wins, so that she wins with probability q and her utility is α , then it necessarily follows that Bob will have utility β . This is because, writing p_A and p_B for the expected payment of Alice and Bob respectively,

$$q + (1 - q)v = W = p_A + p_B + \alpha + \beta.$$

So if indeed $q - p_A = \alpha$, we can conclude that $(1 - q)v - p_B = \beta$ and hence Bob's utility is precisely β . We will therefore focus on Alice's utility for the remainder of the proof.

We can substitute the expressions for β and v (Eq. (1)) into our expression for q to yield

$$q = 2 + \log\left(\frac{\alpha}{1+(e-1)\alpha}\right). \quad (3)$$

This expression is non-decreasing on the interval $[0.27, 0.28]$, so we can conclude (by evaluating the expression on the endpoints) that $q \in [0.3, 0.4]$ for $\alpha \in [0.27, 0.28]$.

The minimum total utility that can be achieved by Alice, while winning with probability q , is if she wins when prices are highest. That is, if she wins whenever the price is at or above $F^{-1}(1-q)$. Under this specification, the utility of Alice would be

$$u_{min} = q - \int_{F^{-1}(1-q)}^v xF'(x)dx.$$

Similarly, the maximum possible utility achievable by Alice is if she wins when prices are lowest; that is, when prices are at or below $F^{-1}(q)$. Under this choice, the utility of Alice would be

$$u_{max} = q - \int_0^{F^{-1}(q)} xF'(x)dx.$$

Since F is continuous on the range $(0, v)$, it is enough to show that $\alpha \in [u_{min}, u_{max}]$, since this implies the existence of an interval upon which Alice could win so that her utility is exactly α .

We next claim that $F(x) = \frac{\beta}{v-x}$ for $x \in [0, F^{-1}(q)]$ and $F(x) = \frac{\alpha}{1-x}$ for $x \in [F^{-1}(1-q), 1]$. To see this, note that since $\beta < v\alpha$ we have that $F(x)$ is precisely $\frac{\beta}{v-x}$ on the range $[0, X]$ and precisely $\frac{\alpha}{1-x}$ on the range $[X, v]$, where X is the solution to $\frac{\alpha}{1-X} = \frac{\beta}{v-X}$. That is, $X = \frac{v\alpha - \beta}{\alpha - \beta}$. For $\alpha \in [0.27, 0.28]$, we can substitute the expressions for β and v (Eq. (1)) into our expression for X to obtain $X = \frac{(e-1)(1-\alpha)}{e}$. This is non-increasing on the interval $[0.27, 0.28]$, so we can conclude (by evaluating on the endpoints) that $X \in [0.45, 0.5]$. Substituting into the definition of F , we conclude that $F(X) \in [0.5, 0.52]$. Since $q \in [0.3, 0.4]$, we have $q < F(X) < 1 - q$. Thus $F^{-1}(q) < X < F^{-1}(1-q)$, which implies the claim.

We can now conclude that $F^{-1}(q) = v - \frac{\beta}{q}$, and hence

$$u_{max} = q - \int_0^{F^{-1}(q)} xF'(x)dx = q - \int_0^{v - \frac{\beta}{q}} \frac{\beta x}{(v-x)^2} dx.$$

Substituting our expressions for v , β (Eq. (1)), and q (Eq. (3)), we find that the upper bound of the integral is equal to

$$1 - \alpha - \frac{\alpha(1 - \alpha)}{(1 + (e-1)\alpha)(2 + \log(\frac{\alpha}{1+(e-1)\alpha}))}.$$

This quantity is increasing in α on the range $[0.27, 0.28]$, so we can plug in $\alpha = 0.28$ to conclude the upper bound on the integral is at most 0.315. We then have

$$u_{max} \geq q - \int_0^{0.315} \frac{\beta x}{(v-x)^2} dx.$$

Again substituting our expressions for v , β , and q , we find that

$$u_{max} \geq 1 - \frac{\alpha(1 - \alpha)(\frac{0.315}{0.685-\alpha} - \log(\frac{1-\alpha}{0.685-\alpha}))}{1 + (e-1)\alpha} + \log(\frac{e\alpha}{1 + (e-1)\alpha}).$$

The expression $\log(\frac{1-\alpha}{0.685-\alpha})$ is at least 0.564 for $\alpha \in [0.27, 0.28]$, so we have

$$u_{max} \geq 1 - \frac{\alpha(1 - \alpha)(\frac{0.315}{0.685-\alpha} - 0.564)}{1 + (e-1)\alpha} + \log(\frac{e\alpha}{1 + (e-1)\alpha}).$$

The derivative of the right-hand side of this inequality is equal to

$$\frac{0.158925 - 0.174385\alpha - 0.753418\alpha^2 + 0.977958\alpha^3 + 0.0676317\alpha^4 - 0.328235\alpha^5}{\alpha(\alpha - 0.685)^2(0.581977 + \alpha)^2},$$

which is positive on the range $\alpha \in [0.27, 0.28]$. We can therefore conclude that our lower bound on u_{max} is non-decreasing, so one can obtain a bound on u_{max} by evaluating at $\alpha = 0.28$, which yields $u_{max} \geq 0.285$. We therefore have that $u_{max} > \alpha$ for each $\alpha \in [0.27, 0.28]$.

We can now turn to u_{min} . We know that $F^{-1}(1 - q) = 1 - \frac{\alpha}{1-q}$, and hence

$$u_{min} = q - \int_{F^{-1}(1-q)}^v xF'(x)dx = q - \int_{1-\frac{\alpha}{1-q}}^v \frac{\alpha x}{(1-x)^2} dx.$$

Using an analysis that closely follows the reasoning above for u_{max} , we can substitute expressions for v and q and conclude that, for $\alpha \in [0.27, 0.28]$, we have $u_{min} \leq 0.12$. So $u_{min} < \alpha$ in this range, and we therefore have $\alpha \in [u_{min}, u_{max}]$ as required. \square

5 Revenue in coarse equilibria

We start with a construction of a two-bidder first-price auction that admits a coarse equilibrium whose revenue is $1 - 2/e$ fraction of the second highest bid.

Lemma 5.1. *There exists a coarse equilibrium of a single-item two-player auction with player values 1 and 1 whose revenue is $1 - 2/e \leq 0.27$.*

Proof. Here is a coarse equilibrium: The two players bid (x, x) where x is distributed according to the cumulative distribution function $F(x) = e^{-1}/(1 - x)$ (for all $0 \leq x \leq 1 - 1/e$), and each of them wins exactly half the time (at each price). Applying the calculation in the previous lemma, the total revenue of this auction is $1 - e^{-1} + e^{-1} \ln e^{-1} = 1 - 2e^{-1}$, and each player's utility is thus e^{-1} . A possible deviation of one of the players to x will yield utility $F(x)(1 - x) = e^{-1}$ and is thus not strictly profitable. Thus we are indeed in a coarse equilibrium. \square

We show that the construction above is essentially the worst possible case across all first price auctions. We first establish this bound for the two-bidder case, then prove the general theorem by reducing an auction with an arbitrary number of bidders and arbitrary values to the two-bidder case. To state this cleanly, we will fix the value of the second highest bidder, Bob, to 1 and let Alice's value v be any quantity that is at least 1.

Lemma 5.2. *Consider a coarse equilibrium of the single-item 2-player first price auction where Bob has value 1 and Alice has value $v \geq 1$. Then, the revenue of the seller is at least $1 - 2/e \geq 0.26$.*

Proof. Let p_1, p_2 be the probabilities that Alice and Bob, respectively, win, and let r_1, r_2 be the total payments, respectively, of Alice and Bob. Clearly $p_1 + p_2 = 1$ and $r_1 + r_2$ is exactly the revenue of the coarse equilibrium which for notational convenience we will denote by $1 - 2\alpha$ where, by way of contradiction, $\alpha > 1/e$. Thus $p_1 + p_2 - r_1 - r_2 = 2\alpha$ and for some $i \in \{1, 2\}$ we have that $p_i - r_i \leq \alpha$. Notice that for $i = 2$ (Bob) $p_2 - r_2$ is exactly Bob's utility, but for $i = 1$, Alice's utility is $vp_1 - r_1$.

Denote by F the cumulative distribution function of the price paid in the equilibrium. If $p_2 - r_2 \leq \alpha$ then Bob will want to deviate from the equilibrium to a fixed bid x whenever $F(x)(1 - x) > \alpha$. It follows that $F(x) \leq \alpha/(1 - x)$, and using the calculation in lemma 2.1 we have that $E_F[x] \geq 1 - \alpha + \alpha \ln \alpha$. But this expectation is exactly the revenue of the auction thus $1 - 2\alpha \geq 1 - \alpha + \alpha \ln \alpha$, which simplifies to $\alpha \leq 1/e$ which is a contradiction to our choice of $\alpha > 1/e$.

We now handle the second case where $p_1 - r_1 \leq \alpha$, in which case we can bound the probability that Alice wins from above by $1 - 1/e$ (since if Bob wins with probability less than $1/e$ then his utility is certainly less than $1/e$ which is already handled by the first case) and thus bound the utility of Alice by $vp_1 - r_1 \leq \alpha + p_1(v - 1) \leq (1 - 1/e)(v - 1) + \alpha$. Alice will prefer deviating to a fixed bid x whenever $F(x)(v - x) > (1 - 1/e)(v - 1) + \alpha$ so we have that $F(x) \leq ((1 - 1/e)(v - 1) + \alpha)/(v - x)$. Using Lemma 2.1 we have that $E_F[x] \geq v - ((1 - 1/e)(v - 1) + \alpha) + ((1 - 1/e)(v - 1) + \alpha) \ln(((1 - 1/e)(v - 1) + \alpha)/v)$. It remains to show that for any $1/e < \alpha \leq 1$ and $v \geq 1$, it holds that $v - ((1 - 1/e)(v - 1) + \alpha) + ((1 - 1/e)(v - 1) + \alpha) \ln(((1 - 1/e)(v - 1) + \alpha)/v) > 1 - 2\alpha$, or equivalently

$$v - ((1 - 1/e)(v - 1) + \alpha) + ((1 - 1/e)(v - 1) + \alpha) \ln(((1 - 1/e)(v - 1) + \alpha)/v) - 1 + 2\alpha > 0, \quad (4)$$

thus deriving a contradiction. For $\alpha = 1/e$, the LHS of Equation 4 is at least 0 (it is exactly 0 for $v = 1$ and increases in v for $v \geq 1$, by standard analysis); therefore, it is sufficient to show that the LHS of Equation 4 is increasing in α . Indeed, the derivative of the LHS is $2 + \ln\left((1 - 1/e) + \frac{\alpha - (1 - 1/e)}{v}\right)$, which equals 1 for $\alpha = 1/e, v = 1$, and is increasing in both α and v . \square

We can now easily conclude the main theorem.

Theorem 5.3. *In every first price auction, with any number of bidders, the revenue in every coarse equilibrium is at least $1 - 2/e \geq 0.26$ fraction of the second highest value.*

Proof. Take an equilibrium of an auction with k bidders with values $v_1 \geq v_2 \geq \dots \geq v_k$. We will now construct an equilibrium of the two-player auction with values $v_1 \geq v_2$ that has the same revenue as does the original auction. After scaling, the main lemma bounds the revenue of the two-player auction to be at least $(1 - 2/e)v_2$ and so this is also the bound on the original one.

To get the coarse equilibrium for the two player auction, simply take the same distribution on bids as in the original auction, but assigning the winning bids of players $i \geq 3$ to one of the first

two bidders (arbitrarily). Notice that since none of the players $i \geq 3$ had a negative utility in the original auction (otherwise they would deviate to 0), and furthermore, each of the first two players gets at least as much utility from winning as do any of the players $i \geq 3$, thus we are only increasing the utilities of the first and second player in the new equilibrium. On the other hand, notice that we have not changed the utilities from deviations at all since these utilities depend only on the distribution of the winning price and not on the identity of the winner. It follows that the first two players still do not want to deviate and so we have a coarse equilibrium in the two-player game. \square

We remark that as the competition increases, the auctioneer's revenue grows. For the case of two symmetric bidders (with value 1), Lemma 5.1 shows a coarse equilibrium with revenue $1 - 2/e$. For the case of n symmetric bidders we show the following.

Theorem 5.4. *In every first price auction, with any number of symmetric bidders with value v , the revenue in every coarse equilibrium is at least $(1 - \frac{n}{e^{n-1}})v$. This is tight.*

Proof. We first show that the revenue is always at least $(1 - \frac{n}{e^{n-1}})v$. Let F be the distribution of the price. The sum of the bidders' utilities is $v - E[x]$ (where x is distributed according to F). Clearly, one of them has utility at most $\frac{1}{n}(v - E[x])$; denote this value by α . Since no deviation to any x is profitable for that player, it holds that $F(x)(v - x) \leq \alpha$ for all x , that is $F(x) \leq \frac{\alpha}{v-x}$. It follows that the expected value of x according to F is at least the expected value of x according to the distribution $\alpha/(v - x)$ which is $v - \alpha + \alpha \ln(\alpha/v)$ (by Lemma 2.1). Substitute $E[x] = v - \alpha n$ (by the definition of α) to get $\alpha \leq v/e^{n-1}$. It follows that $E[x] = v - \alpha n \geq v(1 - \frac{n}{e^{n-1}})$.

We now construct a CCE with revenue at most $(1 - \frac{n}{e^{n-1}})v$. Consider a profile where bidders bid x according to the distribution $F(x) = \alpha/(v - x)$, where $\alpha = v/e^{n-1}$; and each bidder wins with probability $1/n$. The expected payment is $E[x] = v - \alpha + \alpha \ln(\alpha/v)$. The expected utility of a bidder is $1/n(v - E[x])$ and this should be at least α (the deviation utility). Solving for α , we get $\alpha \leq v/e^{n-1}$. So this is an equilibrium, and the revenue is $E[x] = v - \alpha + \alpha \ln(\alpha/v) = v(1 - \frac{n}{e^{n-1}})$. \square

We can also show that as the gap between the highest value and the second highest value increases, the revenue must get close to the second highest value. To state this in the cleanset way, we will fix the value of the second highest bidder, Bob, to 1 and let Alice's value v approach infinity.

Theorem 5.5. *For very $\epsilon > 0$ there exists $v_0 = O(\epsilon^{-4})$ such that in any auction where Alice has value $v \geq v_0$ and Bob has value 1 (and perhaps other players with other values), the revenue is at least $1 - \epsilon$.*

Proof. Assume by way of contradiction that the total revenue is less than $1 - \epsilon$. It follows that with probability of at least $\epsilon/3$ the price paid by the winner is at most $1 - \epsilon/3$ (otherwise the revenue would be bounded below by $(1 - \epsilon/3)^2 \geq 1 - \epsilon$, for small enough ϵ). It follows that Bob must

win the item with probability of at least $\epsilon^2/9$ as otherwise his utility would be less than that while deviating to $1 - \epsilon/3$ would ensure utility of at least that. Now the bound on the revenue implies that the probability that the winning price is very high, greater than $18/\epsilon^2$ can be at most $\epsilon^2/18$. Now consider a deviation of Alice to $18/\epsilon^2$: her probability of winning goes up by at least $\epsilon^2/9 - \epsilon^2/18$ (the probability that Bob wins minus the probability of any bids above $18/\epsilon^2$). Her utility changes as follows: on the up side it increases by at least $\epsilon^2 v/18$ due to the increased winning probability, and on the down side it decreases by at most $18/\epsilon^2$ due to the increased price. The deviation must be beneficial whenever $v > 18^2/\epsilon^4$. \square

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